THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Assignment 1
Due Date: 16 Feb, 2017

1. Let $\gamma(t) = (t, t^2)$, $t \in [0, 2]$ be a curve on $\mathbb{R}^2$. Find the length of the curve.

2. Find the length of the curve defined by the equation $y = f(x) = x^2 + 2x$, $x \in [0, 3]$.

3. Prove that $\sim$ is an equivalence relation on a set A if and only if both of the following hold:

   (a) $a \sim a$ for all $a \in A$;

   (b) let $a, b, c \in A$, if $a \sim b$ and $a \sim c$, then $b \sim c$.

   (Remark: The second statement may be regarded as subsitute of the common notion 1.1 "Things which equal the same thing also equal one another" in Euclid's Elements.)

4. Prove or disprove the following statements of plane geometry ($\mathbb{R}^2$).

   (a) For all distinct points A and B, there exists a straight line L such that A and B lie on the opposite sides of the line L.

   (b) For all distinct points A, B and C, there exists a circle $\mathcal{C}$ such that A, B and C lie on $\mathcal{C}$.

5. Let n be a positive integer and let $\sim$ be a relation defined on $\mathbb{Z}$ which is given by $a \sim b$ if $b - a$ is divisible by n.

   (a) Show that $\sim$ is an equivalence relation.

   (b) Write down the elements of $\mathbb{Z}_n := \mathbb{Z}/\sim$.

   (c) Prove that addition of $\mathbb{Z}$ induces an addition on $\mathbb{Z}_n$.

   (d) Compute $[21] + [35]$ where $[21], [35] \in \mathbb{Z}_6$.

6. Let $\mathcal{P}$ be the set of all line segment on $\mathbb{R}^2$ (in the usual sense) and let $\phi : \mathcal{P} \to \mathbb{R}^+$ be a function such that $\phi(s)$ is the length of the line segment s.

   (a) If s is a line segment on $\mathbb{R}$ with endpoints $(2, 3)$ and $(10, 11)$, find $\phi(s)$.

   (b) Define $\sim$ to be a relation on $\mathcal{P}$ such that $s_1 \sim s_2$ if $\phi(s_1) = \phi(s_2)$, i.e. lengths of s_1 and s_2 are the same. Show that $\sim$ is an equivalence relation on $\mathcal{P}$.

7. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be a distance function defined by $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. With respect to d:

   (a) Find the distance between the points $(1, -2)$ and $(-3, 4)$.

   (b) Draw the circle centered at the origin with distance 1.

   (c) Repeat (a) and (b) by re-defining $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.